Weak solutions to the time-dependent Ginzburg-Landau-Maxwell equations (Regularity and Singularity for Partial Differential Equations with Conservation Laws)

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By

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Abstract

We first prove the existence and uniqueness of global weak solutions to the 3D time-dependent Ginzburg-Landau-Maxwell equations with the Coulomb gauge. Then, we obtain uniform bounds of solutions with respect to the dielectric constant $\epsilon > 0$. Consequently, the existence of global weak solutions to the Ginzburg-Landau equations follows by a compactness argument.

§ 1. Introduction

We consider the 3D time-dependent Ginzburg-Landau-Maxwell system in superconductivity [1, 2]:

$$\eta\partial_t\psi + i\eta\kappa\phi\psi + \left(\frac{i}{\kappa}\nabla + A\right)^2\psi + (|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

$$\epsilon(\partial_t^2 A + \partial_t\nabla\phi) + \partial_t A + \nabla\phi + \mathrm{rot}\,^2 A + \mathrm{Re}\left\{\left(\frac{i}{\kappa}\nabla\psi + \psi A\right)\overline{\psi}\right\} = 0, \tag{1.2}$$

in $Q_T := (0, T) \times \Omega$, with boundary and initial conditions

$$\nabla\psi\cdot\nu = 0,\ A\cdot\nu = 0,\ \mathrm{rot}\,A\times\nu = 0,\ \nabla\phi\cdot\nu = 0 \ \text{ on } \ (0, T)\times\partial\Omega, \tag{1.3}$$

$$(\psi, A, \partial_t A, \phi)(0, \cdot) = (\psi_0, A_0, \tilde{A}_0, \phi_0)(\cdot) \ \text{ in } \ \Omega \subset \mathbb{R}^3. \tag{1.4}$$

Here, the unknowns ψ, A, and ϕ are $\mathbb{C}$-valued, $\mathbb{R}^d$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the

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electric potential, respectively. η and κ are Ginzburg-Landau positive constants, ϵ is the dielectric constant and is supposedly very small in superconductors, and $i := \sqrt{-1}$. Ω is a simply connected and bounded domain with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$. $\overline{\psi}$ denotes the complex conjugate of $\psi, \mathrm{Re}\psi := (\psi + \overline{\psi})/2$ is the real part of ψ and $|\psi|^2 := \psi\overline{\psi}$ is the density of superconductivity carriers. T is any given positive constant.

It is well-known that the Ginzburg-Landau-Maxwell system is gauge invariant, that is, if (ψ, A, ϕ) is a solution of (1.1)-(1.4), then for any real valued smooth function $\chi, (\psi e^{i\kappa\chi}, A + \nabla\chi, \phi - \partial_t\chi)$ is also a solution of (1.1)-(1.4). So in order to obtain the well-posedness of the problem, we need to impose the gauge condition. From physical point of view, one usually has four types of the gauge condition:

(1) Coulomb gauge: $\operatorname{div} A = 0$ in Ω and $\int_\Omega \phi dx = 0$.

(2) Lorentz gauge: $\phi = -\operatorname{div} A$ in Ω.

(3) Lorenz gauge: $\partial_t\phi = -\operatorname{div} A$ in Ω.

(4) Temporal gauge: $\phi = 0$ in Ω.

In 1999, Tsutsumi and Kasai [3] proved the existence and uniqueness of global weak solutions to the problem with the Coulomb gauge under the assumption that $\psi_0 \in H^1 \cap L^\infty, A_0 \in H^1, \tilde{A}_0 \in L^2$, and $\phi_0 \in H^1$.

The first aim of this paper is to generalize the above result under a weaker assumption on the initial data ψ_0. We will prove

Theorem 1.1. *Suppose that* $\psi_0 \in L^\infty \cap W^{\frac{2}{3},\frac{3}{2}}$ *with* $|\psi_0| \le 1, \operatorname{div} A_0 = 0, A_0 \in H^1, \tilde{A}_0 \in L^2$*, and* $\phi_0 \in H^1$*. Then for any* $T > 0$ *there exists a unique weak solution* (ψ, A, ϕ) *of* (1.1)-(1.4) *with Coulomb gauge satisfying*

$$\begin{aligned} & |\psi| \le 1 \ \ in \ \ Q_T, \ \psi \in L^3(0,T;W^{1,3}) \cap L^{\frac{3}{2}}(0,T;W^{2,\frac{3}{2}}), \ \partial_t\psi \in L^{\frac{3}{2}}(Q_T), \\ & A \in L^\infty(0,T;H^1), \ \partial_t A \in L^\infty(0,T;L^2), \\ & \phi \in L^\infty(0,T;H^1), \ \partial_t\phi \in L^2(0,T;H^{-2}). \end{aligned} \tag{1.5}$$

Remark. Our proof is different from that of [3]. Our key estimate is to obtain $W^{2,1}_{3/2}$ estimates of ψ (L^p theory), while their proof in [3] is to get $W^{2,1}_2$ estimates of ψ (Ł^2 theory). Thus our assumption on the initial data ψ_0 is weaker than that in [3].

Remark. When $\psi_0 \in H^1, \operatorname{div} A_0 = 0, A_0 \in H^1, \tilde{A}_0 \in L^2$, and $\phi_0 \in H^1$, we can prove a similar result. Here we do not suppose that $|\psi_0| \le 1$ and the key estimate is to ensure that $\psi \in L^2(0,T;H^2)$, which can be proved easily.

Theorem 1.2. *Suppose that* $\psi_0 \in L^4, \operatorname{div} A_0 = 0, A_0 \in H^1, \tilde{A}_0 \in L^2$*, and* $\phi_0 \in H^1$*. Then for any* $T > 0$ *there exists at least one weak solution* (ψ, A, ϕ) *of* (1.1)-(1.4) *satisfying*

$$\begin{aligned}&\psi \in L^\infty(0,T;L^4)\cap L^6(Q_T)\cap L^2(0,T;H^1), \partial_t\psi \in L^2(0,T;H^{-1}),\\ &A\in L^\infty(0,T;H^1), \partial_t A \in L^\infty(0,T;L^2),\\ &\phi \in L^\infty(0,T;H^1), \partial_t\phi \in L^2(0,T;H^{-2}).\end{aligned} \tag{1.6}$$

Next, we consider the limit as $\epsilon \to 0$. We will prove

Theorem 1.3. *Suppose that* $0<\epsilon<1, \psi_0 \in H^1\cap L^\infty$ *with* $|\psi_0|\le 1, \operatorname{div} A_0 = 0, A_0 \in H^1, \tilde{A}_0 \in L^2$, *and* $\phi_0 \in H^1$. *Then for any* $T>0$ *there exists a unique weak solution* $(\psi_\epsilon, A_\epsilon, \phi_\epsilon)$ *of* (1.1)-(1.4) *satisfying*

$$\begin{aligned}&|\psi_\epsilon|\le 1 \ \ in \ \ Q_T, \ \|\psi_\epsilon\|_{L^\infty(0.T;H^1)}\le C, \ \|\partial_t\psi_\epsilon\|_{L^2(0,T;L^2)}\le C,\\ &\|A_\epsilon\|_{L^\infty(0,T;H^1)}\le C, \ \|\partial_t A_\epsilon\|_{L^2(0,T;L^2)}\le C,\\ &\|\phi_\epsilon\|_{L^2(0,T;H^1)}\le C\end{aligned} \tag{1.7}$$

where C *is independent of* $\epsilon > 0$.

Remark. As soon as the uniform a priori estimates with respect to $\epsilon > 0$ such as (1.7) are established, the standard compactness arguments show the existence of a convergent subsequence $\{\epsilon_j\}\subset(0,1)\to(\psi_{\epsilon_j}, A_{\epsilon_j}, \phi_{\epsilon_j})$ with $\epsilon_1>\epsilon_2>\cdots>\epsilon_j\downarrow 0$ as $j\to\infty$ for (1.1)-(1.4). When $\epsilon=0$, the Ginzburg-Landau-Maxwell system reduces to the well-known Ginzburg-Landau equations, which have received many studies [4, 5, 6, 7, 8, 9, 10, 11, 12].

We will use the notation $W_p^{2,1} := \{\ f \ \big| \ f, \nabla f, \nabla^2 f, \partial_t f \in L^p(\Omega\times(0,T))\}$.

§ 2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To prove the existence part, we only need to prove a priori estimates (1.5).

To begin with, it is easy to infer that [3]:

$$|\psi|\le 1 \ \text{ in } \ Q_T. \tag{2.1}$$

Multiplying (1.1) by $\overline{\psi}$ and integrating parts, then taking the real part, we see that

$$\frac{\eta}{2}\frac{d}{dt}\int|\psi|^2dx+\int\left|\frac{i}{\kappa}\nabla\psi+\psi A\right|^2dx+\int|\psi|^4dx=\int|\psi|^2dx,$$

which gives

$$\int_0^T\int\left|\frac{i}{\kappa}\nabla\psi+\psi A\right|^2dxdt\le C. \tag{2.2}$$

Here and hereafter C will be a constant which may depend on T.

Testing (1.2) by $\partial_t A + \nabla\phi$ and using (2.1) and (2.2), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\epsilon|\partial_t A|^2+\epsilon|\nabla\phi|^2+|\mathrm{rot}\,A|^2)dx+\int(|\partial_t A|^2+|\nabla\phi|^2)dx\\
&=-\int \mathrm{Re}\left\{\left(\frac{i}{\kappa}\nabla\psi+\psi A\right)\overline{\psi}\right\}(\partial_t A+\nabla\phi)dx\\
&\le\left\|\frac{i}{\kappa}\nabla\psi+\psi A\right\|_{L^2}\|\partial_t A+\nabla\phi\|_{L^2}, \qquad (2.3)
\end{aligned}$$

which gives

$$\|\partial_t A\|_{L^\infty(0,T;L^2)}\le C,\ \|\phi\|_{L^\infty(0,T;H^1)}\le C, \qquad (2.4)$$

$$\|A\|_{L^\infty(0,T;H^1)}\le C, \qquad (2.5)$$

where we have used the well-known Poincaré inequalities

$$\|\phi\|_{L^2}\le C\|\nabla\phi\|_{L^2},\ \|A\|_{L^2}\le C\|\mathrm{rot}\,A\|_{L^2}\ \text{ and }\ \|\mathrm{rot}\,A\|_{L^2}=\|\nabla A\|_{L^2}. \qquad (2.6)$$

Inequalities (2.1), (2.2) and (2.5) lead to

$$\|\psi\|_{L^2(0,T;H^1)}\le C. \qquad (2.7)$$

Taking div to (1.2), we deduce that

$$\epsilon\partial_t\Delta\phi+\Delta\phi=-\mathrm{div}\,\mathrm{Re}\left\{\left(\frac{i}{\kappa}\nabla\psi+\psi A\right)\overline{\psi}\right\}. \qquad (2.8)$$

It follows from (2.1), (2.2), (2.4), and (2.8) that

$$\|\partial_t\phi\|_{L^2(0,T;H^{-2})}\le C. \qquad (2.9)$$

Equation (1.1) can be rewritten as

$$\eta\partial_t\psi-\frac{1}{\kappa^2}\Delta\psi=f:=-i\eta\kappa\phi\psi-\frac{2i}{\kappa}A\nabla\psi-|A|^2\psi-(|\psi|^2-1)\psi. \qquad (2.10)$$

By the $W_p^{2,1}$ regularity theory of the heat equation [13], we derive

$$\begin{aligned}
&\int_0^T\|\partial_t\psi\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}dt+\int_0^T\|\psi\|_{W^{2,\frac{3}{2}}}^{\frac{3}{2}}dt\\
&\le C\int_0^T\|f\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}dt+C\|\psi_0\|_{W^{\frac{2}{3},\frac{3}{2}}}^{\frac{3}{2}}\\
&\le C+C\int_0^T\|\phi\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}dt+C\int_0^T\|A\|_{L^6}^{\frac{3}{2}}\|\nabla\psi\|_{L^2}^{\frac{3}{2}}dt\\
&\le C. \qquad (2.11)
\end{aligned}$$

Using the Gagliardo-Nirenberg inequality

$$\|\nabla\psi\|_{L^3}^2 \le C\|\psi\|_{L^\infty}\|\psi\|_{W^{2,\frac{3}{2}}}, \tag{2.12}$$

we get

$$\|\psi\|_{L^3(0,T;W^{1,3})} \le C. \tag{2.13}$$

This completes the proof of (1.5). Now we prove the uniqueness. Let (ψ_j, A_j, ϕ_j) $(j = 1, 2)$ be two weak solutions to (1.1)-(1.4).

We set $\psi := \psi_1 - \psi_2, A := A_1 - A_2$ and $\phi := \phi_1 - \phi_2$. It is straightforward to infer that

$$\begin{aligned} &\eta\partial_t\psi + i\eta\kappa\phi_1\psi + i\eta\kappa\phi\psi_2 - \frac{1}{\kappa^2}\Delta\psi + \frac{2i}{\kappa}A_1\nabla\psi + \frac{2i}{\kappa}A\nabla\psi_2 + A_1^2\psi \\ &\qquad +(A_1 + A_2)A\psi_2 + |\psi_1|^2\psi + (|\psi_1|^2 - |\psi_2|^2)\psi_2 - \psi = 0, \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\epsilon(\partial_t^2 A + \partial_t\nabla\phi) + \partial_t A + \nabla\phi + \mathrm{rot}^2 A \\ &\qquad +\mathrm{Re}\left\{\left(\frac{i}{\kappa}\nabla\psi_1 + \psi_1 A_1\right)\overline{\psi} + \left(\frac{i}{\kappa}\nabla\psi + A_1\psi + A\psi_2\right)\overline{\psi}_2\right\} = 0. \end{aligned} \tag{2.15}$$

Multiplying (2.14) by $\overline{\psi}$, integrating by parts, and taking the real part of the resulting equality, we estimate

$$\begin{aligned} &\frac{\eta}{2}\frac{d}{dt}\int|\psi|^2dx + \frac{1}{\kappa^2}\int|\nabla\psi|^2dx \\ &\le \eta\kappa\int|\phi||\psi||\psi_2|dx + \frac{2}{\kappa}\int|A_1\nabla\psi\overline{\psi}|dx + \frac{2}{\kappa}\int|A\nabla\psi_2\overline{\psi}|dx \\ &\qquad + \int|(A_1 + A_2)A\psi_2\overline{\psi}|dx + \int(|\psi_1| + |\psi_2|)|\psi|^2|\psi_2|dx + \int|\psi|^2dx \\ &\le C\int|\phi||\psi|dx + C\|A_1\|_{L^6}\|\psi\|_{L^3}\|\nabla\psi\|_{L^2} + C\|\nabla\psi_2\|_{L^2}\|A\|_{L^6}\|\psi\|_{L^3} \\ &\qquad +C\|A_1 + A_2\|_{L^6}\|\psi_2\|_{L^\infty}\|A\|_{L^3}\|\psi\|_{L^2} \\ &\qquad +C(\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty})\|\psi_2\|_{L^\infty}\|\psi\|_{L^2}^2 + \|\psi\|_{L^2}^2 \\ &\le C\|\nabla\phi\|_{L^2}^2 + C\|\psi\|_{L^2}^2 + C\|\psi\|_{L^3}\|\nabla\psi\|_{L^2} \\ &\qquad +C\|\nabla\psi_2\|_{L^2}\|A\|_{L^6}\|\psi\|_{L^3} + C\|A\|_{L^3}\|\psi\|_{L^2} \\ &\le C\|\nabla\phi\|_{L^2}^2 + C\|\psi\|_{L^2}^2 + C\|\nabla\psi_2\|_{L^2}^2\|A\|_{H^1}^2 + C\|A\|_{H^1}^2 + \frac{1}{8\kappa^2}\int|\nabla\psi|^2dx. \end{aligned} \tag{2.16}$$

Testing (2.15) by $\partial_t A + \nabla\phi$, using (2.1), (2.5), the Hölder inequality and Young inequality, we have

$$
\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int(|\partial_t A|^2+|\nabla\phi|^2)dx+\int(|\partial_t A|^2+|\nabla\phi|^2)dx+\frac{1}{2}\frac{d}{dt}\int|\mathrm{rot}\,A|^2dx\\
&\le\int\left|\frac{i}{\kappa}\nabla\psi_1+\psi_1A_1\right||\psi||\partial_t A+\nabla\phi|dx\\
&\qquad\qquad+\int\left|\frac{i}{\kappa}\nabla\psi+A_1\psi+A\psi_2\right||\partial_t A+\nabla\phi|dx\\
&\le C\|\nabla\psi_1\|_{L^3}\|\psi\|_{L^6}\|\partial_t A+\nabla\phi\|_{L^2}+C\|A_1\|_{L^6}\|\psi\|_{L^3}\|\partial_t A+\nabla\phi\|_{L^2}\\
&\quad+C(\|\nabla\psi\|_{L^2}+\|A_1\|_{L^3}\|\psi\|_{L^6}+\|A\|_{L^2})\|\partial_t A+\nabla\phi\|_{L^2}\\
&\le\frac{1}{8\kappa^2}\|\nabla\psi\|_{L^2}^2+C(1+\|\nabla\psi_1\|_{L^3}^2)(\|\partial_t A\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2)\\
&\qquad\qquad+C\|\psi\|_{L^2}^2+C\|A\|_{L^2}^2.
\end{aligned}
\tag{2.17}
$$

Summing up (2.16) and (2.17) and using the Gronwall inequality, we arrive at

$$\psi=0, A=0 \quad\text{and}\quad \phi=0.$$

This completes the proof. □

§ 3. Proof of Theorem 1.2

We only need to prove a priori estimates (1.6). Multiplying (1.1) by $|\psi|^2\overline{\psi}$, integrating by parts, and taking the real part of the resulting equality, we observe that

$$\frac{\eta}{4}\frac{d}{dt}\int|\psi|^4dx+\int\left|\frac{i}{\kappa}\nabla\psi+\psi A\right|^2|\psi|^2dx+\int|\psi|^6dx=\int|\psi|^4dx,$$

which gives

$$\psi\in L^\infty(0,T;L^4)\cap L^6(Q_T), \tag{3.1}$$

$$\int_0^T\int\left|\frac{i}{\kappa}\nabla\psi+\psi A\right|^2|\psi|^2dxdt\le C. \tag{3.2}$$

Similarly to (2.3), using (3.2), we still have (2.4) and (2.5).

Finally, we still have (2.7) and (2.9).

This completes the proof. □

§ 4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. Since it has been proved that the problem (1.1)-(1.4) has a unique global weak solution [3], we only need to prove the a priori estimates (1.7). From now on, we drop the subscript ϵ for simplicity. In the following calculations, we need to keep track of the independence of ϵ on constants C.

First, we still have (2.1).

It is well-known that the Ginzburg-Landau free energy given by [2]:

$$G(\psi, A, \phi) := \frac{1}{2}\int\left(\left|\frac{i}{\kappa}\nabla\psi + \psi A\right|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 + |\mathrm{rot}\, A|^2 + \epsilon(|\partial_t A|^2 + |\nabla\phi|^2)\right)dx$$

satisfies

$$\frac{dG}{dt} = -\int(\eta|\partial_t\psi + i\kappa\phi\psi|^2 + |\partial_t A|^2 + |\nabla\phi|^2)dx \leq 0,$$

whence

$$G(\psi, A, \phi) \leq C, \ \int_0^T\int(|\partial_t\psi + i\kappa\phi\psi|^2 + |\partial_t A|^2 + |\nabla\phi|^2)dxdt \leq C. \tag{4.1}$$

Inequalities (2.1), (4.1) and (2.6) easily lead to (1.7). This completes the proof. □

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